(26) 8.5 *Miscellaneous Substitutions*

Expression in integrand	Substitution	EXAMPLE
$\sqrt[n]{x}$, $\sqrt[m]{x}$	$u^{mn} = x$ $mn\,u^{mn-1}\,du = dx$	$I = \displaystyle\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}$ $u^6 = x$ $6\,u^5\,du = dx$ $I = \displaystyle\int \frac{6\,u^5}{\sqrt{u^6} + \sqrt[3]{u^6}}\,du$ $I = \displaystyle\int \frac{6\,u^5}{u^3 + u^2}\,du$
$\sqrt[n]{f(x)}$	$u^n = f(x)$ $n\,u^{n-1}du = f'(x)\,dx$	$I = \displaystyle\int \frac{1}{(x+1)\sqrt{x-2}}\,dx$ $u^2 = x - 2$ $u^2 + 2 = x$ $2u\,du = dx$ $I = \displaystyle\int \frac{2u}{(u^2 + 3)\,u}\,du$ $I = \displaystyle\int \frac{2}{u^2 + 3}\,du$ $I = \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{u}{\sqrt{3}} + c$ $I = \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{\sqrt{x-2}}{\sqrt{3}} + c$ $I = \dfrac{2}{\sqrt{3}} \tan^{-1} \sqrt{\dfrac{x-2}{3}} + c$

Example 1   Evaluate $\displaystyle\int \frac{\coth^3 x}{\sinh^5 x}\,dx$ 37   June   2005

Solution

$$I = \int \frac{\coth^3 x}{\sinh^5 x}\,dx = \int \frac{\cosh^3 x}{\sinh^3 x \cdot \sinh^5 x}\,dx = \int \frac{\cosh^3 x}{\sinh^8 x}\,dx = \int \frac{\cosh^2 x}{\sinh^8 x} \cdot \cosh x\,dx$$

$$= \int \frac{(1 + \sinh^2 x)}{\sinh^8 x} \cdot \cosh x\,dx$$

Let $t = \sinh x$ $\Rightarrow$ $dt = \cosh x\ dx$

$$I = \int \frac{1 + t^2}{t^8}\,dt = \int (t^{-8} + t^{-6})\,dt = \frac{-1}{7}\,t^{-7} - \frac{-1}{5}\,t^{-5} + c = \frac{-1}{7}\,\operatorname{csch}^7 x - \frac{1}{5}\,\operatorname{csch}^5 x + c$$

Example 2   Evaluate $\displaystyle\int \frac{\coth x}{\sqrt{\cosh^2 x - 2}}\,dx$ 43   May   2007

Solution

$$I = \int \frac{\coth x}{\sqrt{\cosh^2 x - 2}}\,dx = \int \frac{\cosh x}{\sinh x \sqrt{\sinh^2 x - 1}}\,dx$$

Let $t = \sinh x$ $\Rightarrow$ $dt = \cosh x\ dx$

$$I = \int \frac{1}{t\sqrt{t^2 - 1}}\,dx = \sec^{-1} t + c = \sec^{-1}(\sinh x) + c$$

Example 3   Evaluate $\displaystyle\int \frac{1}{(e^x + e^{-x})^2}\,dx$ 30   July   2003

Solution

$$I = \int \frac{1}{(e^x + e^{-x})^2}\,dx = \frac{1}{4}\int \frac{4}{(e^x + e^{-x})^2}\,dx = \frac{1}{4}\int \left(\frac{2}{e^x + e^{-x}}\right)^2 dx = \frac{1}{4}\int \operatorname{sec}^2 x\ dx = \frac{1}{4}\tanh x + c$$

Example 4   Evaluate $\displaystyle\int \frac{\sqrt{x} - \sqrt[6]{x}}{\sqrt{x}\left(x - \sqrt[3]{x}\right)}\,dx$

Solution

$$I = \int \frac{\sqrt{x} - \sqrt[6]{x}}{\sqrt{x}\left(x - \sqrt[3]{x}\right)}\,dx$$

$x = u^6$ $\Rightarrow$ $dx = 6u^5\ du$ $\sqrt{x} = u^3$ $\sqrt[6]{x} = u$ $\sqrt[3]{x} = u^2$

$$I = \int \frac{u^3 - u}{u^3(u^6 - u^2)} \cdot 6u^5\ du = 6 \int \frac{u(u^2 - 1)}{u^3 \cdot u^2(u^4 - 1)} \cdot u^5\ du = 6 \int \frac{u(u^2 - 1)}{u^5(u^4 - 1)} \cdot u^5\ du$$

$$= 6 \int \frac{u(u^2 - 1)}{(u^4 - 1)}\,du = 6 \int \frac{u(u^2 - 1)}{(u^2 - 1)(u^2 + 1)}\,du = 6 \int \frac{u}{u^2 + 1}\,du = \frac{6}{2} \int \frac{2u}{u^2 + 1}\,du$$

$$= 3 \ln(u^2 + 1) + c = 3 \ln\left|\sqrt[3]{x} + 1\right| + c$$

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Example 5   Evaluate $\displaystyle\int \frac{2x}{1+\sqrt[3]{3x^2-1}}\,dx$ 41   July   2006

Solution

$3x^2 - 1 = u^3 \qquad \Rightarrow 6x\,dx = 3u^2\,du \qquad \Rightarrow 2x\,dx = u^2\,du$

$u = (3x^2 - 1)^{\frac{1}{3}}$

$I = \displaystyle\int \frac{2x}{1+\sqrt[3]{3x^2-1}}\,dx = \int \frac{u^2}{1+u}\,du$

$I = \displaystyle\int \left(u - 1 + \frac{1}{1+u}\right)\,du$

$\quad = \dfrac{1}{2}u^2 - u + \ln|u+1| + c$

$\quad = \dfrac{1}{2}(3x^2-1)^{\frac{2}{3}} - (3x^2-1)^{\frac{1}{3}} + \ln\left|(3x^2-1)^{\frac{1}{3}}+1\right| + c$

$$
\begin{array}{r}
u-1 \\
u+1\,\overline{\smash{\big)}\,u^2 } \\
\underline{-} \\
u^2 + u \\
\underline{-u} \\
\overset{+}{-}u \overset{+}{-}1 \\
\underline{} \\
+1
\end{array}
$$

Example 6   Evaluate $\displaystyle\int \frac{1}{\sqrt{2+\sqrt{x-1}}}\,dx$

Solution

$u^2 = 2 + \sqrt{x-1} \qquad \Rightarrow \sqrt{x-1} = u^2 - 2 \qquad \Rightarrow x - 1 = u^4 - 2u^2 + 4$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad dx = (4u^3 - 4u)\,du$

$I = \displaystyle\int \frac{1}{\sqrt{2+\sqrt{x-1}}}\,dx = \int \frac{4u^3 - 4u}{u}\,du = \int (4u^2 - 4)\,du = \frac{4}{3}u^3 - 4u + c$

$\qquad\qquad = \dfrac{4}{3}\left(\sqrt{2+\sqrt{x-1}}\right)^3 - 4\sqrt{2+\sqrt{x-1}} + c$

Example 7   Evaluate $\displaystyle\int \frac{\sqrt{\ln|2x+4|+7}}{x+2}\,dx$

Solution

$t = \ln|2x+4| + 7 \qquad \Rightarrow dt = \dfrac{2}{2x+4}\,dx \qquad \Rightarrow dt = \dfrac{1}{x+2}\,dx$

$I = \displaystyle\int \frac{\sqrt{\ln|2x+4|+7}}{x+2}\,dx = \int \sqrt{t}\,dt = \int t^{\frac{1}{2}}\,dt = \frac{2}{3}t^{\frac{3}{2}} + c = \frac{2}{3}(\ln|2x+4|+7)t^{\frac{3}{2}} + c$

Example 8   Evaluate $\int \tanh(\ln x)\, dx$ 2   May   1995

Solution

$$I = \int \tanh(\ln x)\, dx = \int \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}\, dx = \int \frac{x - \frac{1}{x}}{x + \frac{1}{x}}\, dx = \int \frac{x^2 - 1}{x^2 + 1}\, dx$$

$$= \int \frac{x^2 + 1 - 2}{x^2 + 1}\, dx = \int \left(\frac{x^2 + 1}{x^2 + 1} - \frac{2}{x^2 + 1} \right) dx = \int \left(1 - \frac{2}{x^2 + 1} \right) dx = x - 2\tan^{-1} x + c$$

Example 9   Evaluate $\int \frac{x^{15}}{x^{32} - 16}\, dx$

Solution

$$I = \int \frac{x^{15}}{x^{32} - 16}\, dx$$

$$u = x^{16} \qquad \Rightarrow du = 16\, x^{15}\, dx \qquad \Rightarrow \frac{1}{16} du = x^{15}\, dx$$

$$I = \frac{1}{16} \int \frac{1}{u^2 - 16}\, dx = \frac{1}{16} \int \frac{1}{(u - 4)(u + 4)}\, dx$$

$$\frac{1}{(u - 4)(u + 4)} = \frac{A}{u - 4} + \frac{B}{u + 4}$$

$$A(u + 4) + B(u - 4) = 1$$

at $u = -4$ $\qquad \Rightarrow \quad B(-8) = 1$ $\qquad \Rightarrow \boxed{B = \frac{-1}{8}}$

at $u = 4$ $\qquad \Rightarrow \qquad A(8) = 1$ $\qquad \Rightarrow A = \frac{1}{8}$

$$I = \frac{1}{16} \int \frac{\frac{1}{8}}{(u - 4)} - \frac{\frac{1}{8}}{(u + 4)}\, du = \frac{1}{128} \ln|u - 4| - \frac{1}{128} \ln|u + 4| + c$$

$$= \frac{1}{128} \ln|x^{16} - 4| - \frac{1}{128} \ln|x^{16} + 4| + c$$

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Homework

1	Evaluate the integral	$\int \dfrac{\coth^3 x}{\sinh^5 x}\, dx$	
2	Evaluate the integral	$\int \dfrac{dx}{\sqrt{1 + \sinh^2 x}}$	
3	Evaluate the integral	$\int \dfrac{\coth x}{\sqrt{\cosh^2 x - 2}}\, dx$	
4	Evaluate the integral	$\int \dfrac{\cosh x}{\sqrt{1 - \sinh^2 x}}\, dx$	
5	Evaluate the integral	$\int \dfrac{1}{(e^x + e^{-x})^2}\, dx$	
6	Evaluate the integral	$\int \dfrac{\cosh x}{\cosh^2 x + 2 \sinh x + 1}\, dx$	
7	Evaluate the integral	$\int \dfrac{1}{e^x - 2 + e^{-x}}\, dx$	
8	Evaluate the integral	$\int \dfrac{\sqrt{x} - \sqrt[6]{x}}{\sqrt{x}\left(x + 4\sqrt[3]{x}\right)}\, dx$	42 December 2006
9	Evaluate the integral	$\int \dfrac{\sqrt[3]{x}}{\sqrt{x} + \sqrt[6]{x}}\, dx$	
10	Evaluate the integral	$\int \dfrac{2x}{1 + \sqrt[3]{3x^2 - 1}}\, dx$	
11	Evaluate the integral	$\int \dfrac{x}{\sqrt[3]{x - 2}}\, dx$	
12	Evaluate the integral	$\int \dfrac{1}{\sqrt{1 - \sqrt{x - 2}}}\, dx$	5 May 1996
13	Evaluate the integral	$\int \sqrt{1 + \sqrt{x}}\, dx$	32 December 2003
14	Evaluate the integral	$\int \dfrac{x^5}{\sqrt{1 - x^3}}\, dx$	

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Homework

15	Evaluate the integral	$\int \dfrac{1}{\sqrt[3]{x}\sqrt{1+\sqrt[3]{x}}}\,dx$	6 July 1996
16	Evaluate the integral	$\int \dfrac{1-\sinh x}{\sqrt{1+\sinh^2 x}}\,dx$	8 May 1997
17	Evaluate the integral	$\int \sqrt{1-2\sqrt{x}}\,dx$	10 August 1997
18	Evaluate the integral	$\int \dfrac{1}{x(\sqrt{x}+\sqrt[6]{x})}\,dx$	14 November 1998
19	Evaluate the integral	$\int \sqrt{1+\sqrt[3]{x}}\,dx$	12 May 2001
20	Evaluate the integral	$\int \dfrac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$	19 May 2000
21	Evaluate the integral	$\int \dfrac{\sqrt[3]{x}}{\sqrt{x}(\sqrt[3]{x}+4)}\,dx$	24 July 2001
22	Evaluate the integral	$\int \dfrac{1}{\sqrt[3]{x}-\sqrt{x}}\,dx$	25 December 2001
23	Evaluate the integral	$\int \dfrac{1}{x+\sqrt[4]{x}}\,dx$	28 May 2003
24	Evaluate the integral	$\int \dfrac{1}{\sqrt{2-\sqrt{x}}}\,dx$	29 May 2003
25	Evaluate the integral	$\int \dfrac{\sqrt{x}}{1-\sqrt{1-\sqrt{x}}}\,dx$	30 July 2003
26	Evaluate the integral	$\int \dfrac{1}{\sqrt{x}\sqrt{1-\sqrt[3]{x}}}\,dx$	39 December 2005
27	Evaluate the integral	$\int \dfrac{\sqrt[4]{x}}{x(\sqrt[3]{x}+\sqrt{x})}\,dx$	45 December 2007
28	Evaluate the integral	$\int \dfrac{1}{x\sqrt{\sqrt{x}-4}}\,dx$	46 July 2008

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Homework

29	Evaluate the integral	$\displaystyle\int \frac{x^{\frac{1}{2}} - 1}{x\left(x^{\frac{1}{6}} + x^{\frac{1}{2}}\right)}\, dx$	47	December 2008
30	Evaluate the following integral : $(3\frac{1}{2}$ points $)$	$\displaystyle\int \frac{1}{\sqrt{x} + 2\sqrt[4]{x} + 2}\, dx$	50	Dec. 15, 2009
31	Evaluate the following. [3.5 pts.]	$\displaystyle\int \frac{x}{1 + \sqrt{2 + x}}\, dx$	51	May 13, 2010
32	Evaluate the following integral [3 marks e]	$\displaystyle\int \frac{1}{\sqrt{1 - e^{-x}}}\, dx$	52	July 24, 2010
33	(3 pts.) Evaluate the following integral	$\displaystyle\int \frac{dx}{(1 - x^{1/3})\, x^{1/2}}$	53	11 Dec. 2010
34	Evaluate the following integral	$\displaystyle\int \sinh^3 x \cosh^2 x \, dx$	54	12/05/2011
35	Evaluate the following integral	$\displaystyle\int \frac{1}{x\left(1 + \sqrt[3]{x}\right)^2}\, dx$	54	12/05/2011
36	Evaluate the following integral	$\displaystyle\int \frac{\sinh x}{2\sinh^2 x + 3\cosh x}\, dx$	35	January 24, 2010
37	Evaluate the following integral	$\displaystyle\int \frac{1}{3\cosh x + 4\sinh x + 3}\, dx$	37	August 7, 2010

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